Basic Mathematics

Changing Units

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The aim of this package is to provide a short self assessment programme for students who wish to learn how to convert between different units.

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1. Introduction

The use of different units in science and everyday life makes it important to be able to **convert between different units.**

Example 1 One kilometre is roughly $\frac{5}{8}$ of a mile. What is a mile in kilometres?

$$\begin{aligned} 1\,\text{km} &= \frac{5}{8}\,\text{mile} && (\textit{multiply by } 8) \\ 8\,\text{km} &= 5\,\text{miles} && (\textit{divide by } 5) \\ \frac{8}{5}\,\text{km} &= 1\,\text{mile} \end{aligned}$$

Therefore one mile is roughly $\frac{8}{5}$ of a kilometre.

Exercise 1. Express the following quantities in the units requested (click on the **green** letters for the solutions).

(a) One year in seconds.

(b) 0.016 miles in kilometres.

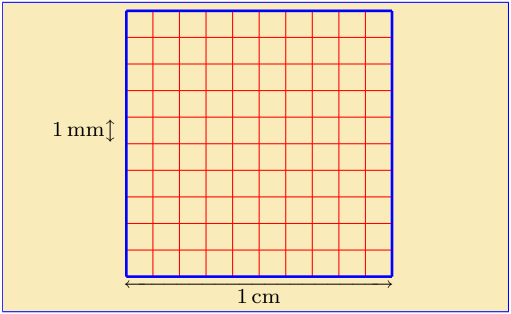
(c) A speed of 10 miles per hour in kilometres per hour.

(d) One million pounds in pennies.

2. Powers of Units

It is important to take care with powers of units.

Example 2 Consider the area of the square drawn below. From the package on **Dimensional Analysis** we know that this has dimensions of length squared (L^2). Its numerical value depends on the units used.



Two possible ways of expressing the area are **either**:

$(1\,\mathrm{cm}) \times (1\,\mathrm{cm}) = 1\,\mathrm{cm}^2$ **or** $(10\,\mathrm{mm}) \times (10\,\mathrm{mm}) = 100\,\mathrm{mm}^2$.

In general, different units are linked by a **conversion factor** and an area given in one unit may be expressed in another unit by multiplying the area by the square of the conversion factor.

Since $1\,\mathrm{cm} = 10\,\mathrm{mm}$, the conversion factor is 10. So an area of one square centimetre is $10^2 = 100$ square millimetres.

Now try these two short quizzes:

Quiz Which of the following is the area of the square whose sides are 1 cm long in the SI units of square metres?

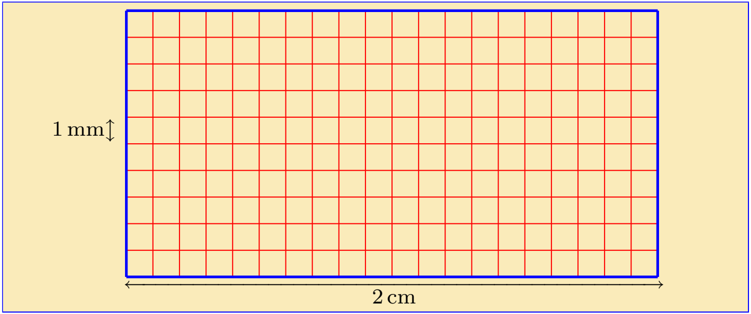
(a) $0.01\,\mathrm{m}^2$ (b) $100\,\mathrm{m}^2$ (c) $0.0001\,\mathrm{m}^2$ (d) $10,000\,\mathrm{m}^2$

Quiz An area of a square mile is a mile times a mile. Recalling that $8\,\mathrm{km} \approx 5\,\mathrm{miles}$, what is this in square kilometres?

(a) $\frac{25}{64}\,\mathrm{km}^2$ (b) $\frac{64}{25}\,\mathrm{km}^2$ (c) $\frac{16}{10}\,\mathrm{km}^2$ (d) $\frac{10}{16}\,\mathrm{km}^2$

Note that we multiply the area by the square of the conversion factor. See also the following example.

Example 3 Consider the area of the rectangle drawn below.



Its area is given by the product of the sides. This is **either**:

$$\text{area} = (2\,\text{cm}) \times (1\,\text{cm}) = 2\,\text{cm}^2$$

or

$$\text{area} = (20\,\text{mm}) \times (10\,\text{mm}) = 2 \times 10^2\,\text{mm}^2 = 200\,\text{mm}^2$$

Note the square of the conversion factor in the last line.

In general an area of $A\,\mathrm{cm}^2$ can be expressed in square millimetres as

$$\begin{aligned} A\,\mathrm{cm}^2 &= A \times (10\,\mathrm{mm})^2 \\ &= A \times 10^2\,\mathrm{mm}^2 \\ &= 100A\,\mathrm{mm}^2 \end{aligned}$$

Similarly a volume (whose dimensions are L^3) can be converted from one set of units to another by multiplying by the cube of the appropriate conversion factor.

Quiz What is a volume of $3\,\mathrm{cm}^3$ expressed in cubic millimetres?

(a) $300\,\mathrm{mm}^3$ (b) $27,000\,\mathrm{mm}^3$ (c) $3 \times 10^4\,\mathrm{mm}^3$ (d) $3,000\,\mathrm{mm}^3$

Quiz A litre (l) is a cubic decimetre, where $1\,\mathrm{dm} = 0.1\,\mathrm{m}$. Express $0.5\,\mathrm{l}$ in cubic metres.

(a) $5 \times 10^{-4}\,\mathrm{m}^3$ (b) $1.25 \times 10^{-2}\,\mathrm{m}^3$ (c) $0.005\,\mathrm{m}^3$ (d) $5,000\,\mathrm{m}^3$

Similarly, with negative powers of a unit we divide by the appropriate power of the conversion factor. An example of this follows.

Example 4 To convert a density (dimension ML^{-3}) of $1\,\mathrm{kg\,m}^{-3}$ into $\mathrm{kg\,dm}^{-3}$ we use

$$\begin{aligned}1\,\mathrm{kg\,m}^{-3} &= 1\,\frac{\mathrm{kg}}{\mathrm{m}^3}\\ &= 1\,\frac{\mathrm{kg}}{(10\mathrm{dm})^3}\\ &= 1\,\frac{\mathrm{kg}}{1000\mathrm{dm}^3}\\ &= 0.001\,\mathrm{kg\,dm}^{-3}\end{aligned}$$

This may now be practised in the next quiz:

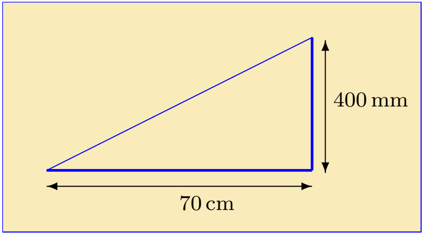
Quiz Express a density of $3 \times 10^2\,\mathrm{kg\,m}^{-3}$ in terms of kilograms per cubic centimetre.

(a) $3 \times 10^{-4}\,\mathrm{kg\,cm}^{-3}$ (b) $2.7 \times 10^{11}\,\mathrm{kg\,cm}^{-3}$
(c) $3 \times 10^{8}\,\mathrm{kg\,cm}^{-3}$ (d) $3 \times 10^{5}\,\mathrm{kg\,cm}^{-3}$

Exercise 2. Perform the following unit conversions (click on the **green** letters for the solutions).

(a) Find the density $3 \times 10^2\,\mathrm{kg\,m^{-3}}$ in grams per cubic decimetre.

(b) Calculate the area of the triangle below in square metres.



(c) The power in Watts of a device that uses $60\,\mathrm{mJ}$ (milli Joules) of energy in half a microsecond. (Power is the rate of conversion of energy with time. See the package on **Units**.)

(d) A kilowatt hour is the energy used by a device with a power output of one kilowatt in one hour. Express this in Joules.

3. Equations and Units

Generally it is best in equations to express all quantities in SI units before performing calculations.

Example 4 The electron Volt (eV) is a widespread unit of energy in atomic and sub-atomic physics. It is related to the SI unit of energy, the Joule, by: $1\,\mathrm{eV} = 1.6\times 10^{-19}\,\mathrm{J}$. The energy of a photon is related to its frequency ν by $E = h\nu$, where $h = 6.6\times 10^{-34}\mathrm{Js}$. What is the frequency of a photon with energy $2.2\,\mathrm{eV}$?

The energy of the photon in SI units is $E = 2.2\times 1.6\times 10^{-19} = 3.52\times 10^{-19}\,\mathrm{J}$. It follows that the photon's frequency is

$$\begin{aligned} \nu &= \frac{E}{h} \\ &= \frac{3.52\times 10^{-19}\,\mathrm{J}}{6.6\times 10^{-34}\mathrm{Js}} \\ &= \frac{3.52}{6.6}\times 10^{-19+34}\,\mathrm{s}^{-1} \\ \therefore \nu &= 5.3\times 10^{14}\,\mathrm{s}^{-1} \end{aligned}$$

EXERCISE 3. Put the quantities below into SI units to perform the requested calculation (click on the **green** letters for the solutions).

(a) Ohm's law $V = IR$ links the voltage V across a resistance R to the current I flowing through it. Calculate the voltage across a $4.3\,\Omega$ (Ohm) resistor if a $4\,\mathrm{mA}$ (milli-Amp) current is measured.

(b) The escape speed v_{esc} of a projectile from a planet is given by $v_{\mathrm{esc}} = \sqrt{2gr}$ where r is the radius of the planet and g is the acceleration due to gravity on the surface of the planet. The Earth's radius is $6380\,\mathrm{km}$ and $g = 9.8\,\mathrm{m\,s^{-2}}$. Calculate the escape speed from the Earth.

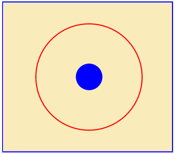
(c) Air pressure P at sea level is roughly $100\,\mathrm{kPa}$ (kilo-Pascal). The height H of a mercury column in a barometer is related to the pressure by $P = H\rho g$ where ρ is the density of Mercury (14 Tonnes per cubic metre) and g is the acceleration due to gravity. Find the height of the column of Mercury.

Here are two further quizzes to practise on:

Quiz The energy E required to melt a mass m of a substance is given by $E = m\ell$ where ℓ is the specific latent heat of fusion (in $\mathrm{J\,kg^{-1}}$). If $5\,\mathrm{MJ}$ is required to melt $2 \times 10^4\mathrm{g}$ of a solid, what is ℓ?

(a) $0.2\mathrm{J\,kg^{-1}}$ (b) $250{,}000\mathrm{J\,kg^{-1}}$ (c) $2.5\mathrm{J\,kg^{-1}}$ (d) $20\mathrm{J\,kg^{-1}}$

Quiz The area of a circle is πr^2 where r is its radius. Calculate the area of the region between the two circles below if the larger one has an area of $1.69\pi\,\mathrm{m}^2$ and the smaller one has a radius of $50\,\mathrm{cm}$.



(a) $1.44\pi\,\mathrm{m}^2$ (b) $0.194\pi\,\mathrm{m}^2$ (c) $2.5\pi\,\mathrm{m}^2$ (d) $1.44 \times 10^{-3}\pi\,\mathrm{m}^2$

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. The sunspot cycle takes place over 11 years. Roughly how many solar cycles take place per millennium?

(a) 110 (b) 11 (c) 91 (d) 181

2. What is a cubic decimetre in terms of cubic centimetres?

(a) $1 \times 10^{-3}\,\mathrm{cm}^3$ (b) $1 \times 10^{4}\,\mathrm{cm}^3$

(c) $1 \times 10^{6}\,\mathrm{cm}^3$ (d) $1 \times 10^{3}\,\mathrm{cm}^3$

3. Express the area of a square of side $5\,\mu\mathrm{m}$ in square metres.

(a) $25\,\mu\mathrm{m}^2$ (b) $2.5 \times 10^{-12}\,\mathrm{m}^2$

(c) $2.5 \times 10^{-11}\,\mathrm{m}^2$ (d) $2.5 \times 10^{-13}\,\mathrm{m}^2$

4. Find the current in Amperes through a $10\,\Omega$ resistor with a $2\,\mathrm{mV}$ potential difference across it.

(a) $2 \times 10^{-3}\,\mathrm{A}$ (b) $500\,\mathrm{A}$ (c) $0.2\,\mathrm{A}$ (d) $2 \times 10^{-4}\,\mathrm{A}$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) To calculate how many seconds there are in a year, recall that there are 365 days each of which lasts 24 hours. In each hour there are 60 minutes each of 60 seconds duration. Thus we get

$$\begin{aligned}\text{No. of seconds in a year} &= 365 \times 24 \times 60 \times 60 \\ &= 31,536,000\,\mathrm{s}\end{aligned}$$

There are 3.1536×10^6 seconds in a year. (Just over 30 million seconds.)

Click on the **green** square to return

□

Exercise 1(b) The distance 0.016 miles is

$$\begin{aligned} 0.016\,\text{miles} &= 0.016 \times \frac{8}{5}\,\text{km} \\ &= 0.0256\,\text{miles} \end{aligned}$$

Click on the **green** square to return

□

Exercise 1(c) A speed of 10 miles per hour is

$$\begin{aligned} 10\,\text{miles per hour} &= \frac{10\,\text{miles}}{1\,\text{hour}} \\ &= \frac{10 \times \frac{8}{5}\,\text{kilometres}}{1\,\text{hour}} \\ &= 16\,\text{kilometres per hour} \end{aligned}$$

Click on the **green** square to return □

Exercise 1(d) One million pounds in pennies is given by

$$10^6 \times 100 = 10^8 \text{ pennies}$$

which is just one hundred million pennies.

Click on the **green** square to return □

Exercise 2(a) To calculate the density $3 \times 10^2\,\mathrm{kg\,m^{-3}}$ in grams per cubic decimetre. We need to use the conversion factors:

$$\begin{aligned} 1\,\mathrm{kg} &= 10^3\,\mathrm{g} \\ 1\,\mathrm{m} &= 10\,\mathrm{dm} \end{aligned}$$

So we have

$$\begin{aligned} 3 \times 10^2\,\mathrm{kg\,m^{-3}} &= 3 \times 10^2 \times \frac{10^3}{(10)^3} \\ &= 3 \times 10^2\,\mathrm{g\,dm^{-3}} \end{aligned}$$

Click on the **green** square to return □

Exercise 2(b) We need to calculate the area of the triangle with base 70 cm and height 400 mm. In SI units these lengths are 0.7 m and 0.4 m respectively.

The formula for the area of a triangle is

$$\text{area} = \frac{1}{2}\text{base} \times \text{height}$$

So the area is $\frac{1}{2} \times 0.7 \times 0.4 = 0.14\,\text{m}^2$.

Click on the **green** square to return

$\square$

Exercise 2(c) We want to find the power in Watts of a device that uses 60 mJ (milli Joules) of energy in half a microsecond. In SI units

$$\begin{aligned} 60\,\mathrm{mJ} &= 60 \times 10^{-3}\,\mathrm{J} = 6 \times 10^{-2}\,\mathrm{J} \\ 0.5\,\mu\mathrm{s} &= 0.5 \times 10^{-6}\,\mathrm{s} = 5 \times 10^{-7}\,\mathrm{s} \end{aligned}$$

We recall that

$$\text{Power} = \frac{\text{energy}}{\text{time}}$$

so the power is

$$\begin{aligned} \text{Power} &= \frac{6 \times 10^{-2}\,(\mathrm{J})}{5 \times 10^{-7}\,(\mathrm{s})} \\ &= 1.2 \times 10^{-2+7}\,\mathrm{J\,s^{-1}} \\ &= 1.2 \times 10^{5}\,\mathrm{W} \end{aligned}$$

where we recall that a Watt is a Joule per second.

Click on the **green** square to return

□

Exercise 2(d) To express a kilowatt hour in Joules, note that a kilowatt hour is the energy used in one hour by a device with a power consumption. In SI units an hour is $3,600\,\mathrm{s}$ and a kilowatt is $1,000\,\mathrm{W}$. So we have

$$\text{energy} = 1,000 \times 3,600 = 3.6 \times 10^6\,\mathrm{J}$$

where we again use that $1\,\mathrm{J} = 1\,\mathrm{W\,s}$.

Click on the **green** square to return

□

Exercise 3(a) We want to find the voltage across a $4.3\,\Omega$ resistor if a $4\,\mathrm{mA}$ (milli-Amp) current is measured. In SI units the current is $I = 4 \times 10^{-3}\,\mathrm{A}$ and the resistance is $R = 4.3\,\Omega$. From Ohm's law, $V = IR$, so

$$V = 4 \times 10^{-3}\,(\mathrm{A}) \times 4.3\,\Omega = 1.72 \times 10^{2}\,\mathrm{V}$$

where we use that $1\mathrm{A}\,\Omega = 1\,\mathrm{V}$.

Click on the **green** square to return

□

Exercise 3(b) We need to find the escape speed of a projectile from the Earth: $v_{\text{esc}} = \sqrt{2gr}$.
In SI units $g = 9.8\,\text{m}\,\text{s}^{-2}$ and $r = 6380 \times 1000 = 6.38 \times 10^6\,\text{m}$. Substituting these values into the equation gives:

$$\begin{aligned}
v_{\text{esc}} &= \sqrt{2 \times 9.8\,\text{m}\,\text{s}^{-2} \times 6.38 \times 10^6\,\text{m}} \\
&= \sqrt{1.25 \times 10^8 \text{m}^2\,\text{s}^{-2}} \\
&= 1.1 \times 10^4\,\text{m}\,\text{s}^{-1}
\end{aligned}$$

Click on the **green** square to return

□

Exercise 3(c) We wish to calculate the height of a column of Mercury in a barometer if the air pressure, P, is 100 kPa. We use

$$P = H\rho g \qquad \text{or} \quad H = \frac{P}{\rho g}$$

where $g = 9.8\,\mathrm{m\,s^{-2}}$ and $\rho = 14$ Tonnes per cubic metre. In SI units $\rho = 14 \times 1000 = 1.4 \times 10^4\,\mathrm{kg\,m^{-3}}$ and $P = 100 \times 1000 = 10^5\,\mathrm{Pa}$. This gives

$$\begin{aligned} H &= \frac{10^5\,\mathrm{Pa}}{1.4 \times 10^4\,\mathrm{kg\,m^{-3}} \times 9.8\,\mathrm{m\,s^{-2}}} \\ &= \frac{1}{1.4 \times 9.8} \times 10^{5-4} \times \mathrm{N\,m^{-2}kg^{-1}m^{3}m^{-1}s^{2}} \\ &= 0.73\,\mathrm{N\,kg^{-1}\,s^{2}} \end{aligned}$$

Now $1\,\mathrm{N} = 1\,\mathrm{kg\,ms^{-2}}$, so therefore the height is $H = 0.73\,\mathrm{m}$.

Click on the **green** square to return □

Solutions to Quizzes

Solution to Quiz: We want to express an area of $1\,\mathrm{cm}^2$ in square metres.

One metre is $100\,\mathrm{cm}$, so $1\,\mathrm{cm} = 10^{-2}\,\mathrm{m}$. Thus the conversion factor is 10^{-2}.

$$\begin{aligned} \therefore 1\,\mathrm{cm}^2 &= (10^{-2})^2\,\mathrm{m}^2 \\ &= 1 \times 10^{-4}\,\mathrm{m}^2 \end{aligned}$$

A square centimetre is a ten thousandth part of a square metre.

End Quiz

Solution to Quiz: We want to express an area of $1\,\mathrm{mile}^2$ in square kilometres.

One mile is $\frac{8}{5}\,\mathrm{km}$, so the conversion factor is $\frac{8}{5}$. We thus obtain

$$\begin{aligned} 1\,\mathrm{mile}^2 &= \left(\frac{8}{5}\right)^2\,\mathrm{km}^2 \\ &= \frac{64}{25}\,\mathrm{km}^2 \end{aligned}$$

End Quiz

Solution to Quiz: We want to express a volume of $3\,\text{cm}^3$ in cubic millimetres.

One centimetre is $10\,\text{mm}$, so the conversion factor is 10. We thus obtain

$$\begin{aligned} 3\,\text{cm}^3 &= 3 \times (10)^3\,\text{mm}^3 \\ &= 3,000\,\text{cm}^3 \end{aligned}$$

End Quiz

Solution to Quiz: We want to express a volume of 0.5 l in cubic metres. A litre is $1\,\mathrm{dm}^3$ and $1\,\mathrm{dm} = 0.1\,\mathrm{m}$, i.e., the conversion factor is 0.1. Thus

$$\begin{aligned} 0.5\,\mathrm{l} &= 0.5\,\mathrm{dm}^3 \\ &= 0.5 \times (0.1)^3\,\mathrm{m}^3 \\ &= 5 \times 10^{-1} \times 10^{-3}\,\mathrm{m}^3 \\ &= 5 \times 10^{-4}\,\mathrm{m}^3 \end{aligned}$$

End Quiz

Solution to Quiz: We have to express a density of $3 \times 10^2\,\mathrm{kg\,m^{-3}}$ in terms of kilograms per cubic centimetre.

A metre is $100\,\mathrm{cm}$, so the conversion factor is 100. One cubic metre is thus $(100)^3 = 10^6\,\mathrm{cm}^3$. Therefore

$$\begin{aligned}
3 \times 10^2\,\mathrm{kg\,m^{-3}} &= \frac{3 \times 10^2\,\mathrm{kg}}{1\,\mathrm{m}^3} \\
&= \frac{3 \times 10^2\,\mathrm{kg}}{1 \times 10^6\,\mathrm{cm}^3} \\
&= 3 \times 10^2 \times 10^{-6}\,\mathrm{kg\,cm^{-3}} \\
&= 3 \times 10^{-4}\,\mathrm{kg\,cm^{-3}}
\end{aligned}$$

End Quiz

Solution to Quiz: We are given $E = m\ell$ and need to calculate ℓ given that $E = 5\,\mathrm{MJ}$ and $m = 2 \times 10^4$g.
Rearranging the equation we obtain: $\ell = E/m$. We also need to express E and m in SI units:

$$\begin{aligned} E &= 5 \times 10^6\,\mathrm{J} \\ m &= 2 \times 10^4 \times 10^{-3}\,\mathrm{kg} = 20\,\mathrm{kg} \end{aligned}$$

Thus we have

$$\ell = \frac{E}{m} = \frac{5 \times 10^6\,(\mathrm{J})}{20\,(\mathrm{kg})} = 2.5 \times 10^5\,\mathrm{J\,kg^{-1}}$$

End Quiz

Solution to Quiz: The area of the disk between the two circles is given by the difference of their areas. The larger has an area of $1.69\pi\,\mathrm{m}^2$ while the smaller has a radius of $50\,\mathrm{cm}$. In SI units the radius is $0.5\,\mathrm{m}$. Such a circle has area $\pi r^2 = \pi(0.5)^2\,\mathrm{m}^2 = 0.25\pi\,\mathrm{m}^2$. Thus the difference is

$$\text{area of difference} = 1.69\pi - 0.25\pi = 1.44\pi\,\mathrm{m}^2$$

End Quiz